CCFU Proof 21

Higher Alternating Closure

Given. $W = S_1 \cap S_2$, $\dim W = 3$ [Proof 20].

Step 1 — The higher closure of W. Define:

$$V_W = \mathbb{R} \cdot \tau \oplus W \oplus W^*, \qquad \dim V_W = 1 + 3 + 3 = 7.$$

W^* is the linear dual of W. τ is a formal line. ■

Step 2 — The natural 3-forms on V_W. $\Lambda^3(V_W^*)$ decomposes under $\mathrm{SL}(W)$ as:

$$\Lambda^3 W^* \oplus \Lambda^3 W \oplus (\mathbb{R} \otimes \Lambda^2 W^*) \oplus (\mathbb{R} \otimes \Lambda^2 W) \oplus (\mathbb{R} \otimes W^* \otimes W) \oplus (\Lambda^2 W^* \otimes W) \oplus (W^* \otimes \Lambda^2 W).$$

$\mathrm{SL}(W)$-invariant 3-forms in each component:

- $\Lambda^3 W^*$: dim 1 $\to$ vol_W (volume form of W)
- $\Lambda^3 W$: dim 1 $\to$ vol_{W^*} (volume form of W^*)
- $\mathbb{R} \otimes W^* \otimes W$: dim 1 $\to$ $\tau \wedge \mathrm{ev}$ (evaluation pairing)
- All others: dim 0.

Total $\mathrm{SL}(W)$-invariant 3-forms: exactly 3. Basis: $\{\tau \wedge \mathrm{ev},\ \mathrm{vol}_W,\ \mathrm{vol}_{W^*}\}$.

Note. $\tau \wedge \mathrm{ev}$ is $\mathrm{GL}(W)$-invariant (canonical). vol_W and vol_{W^*} are $\mathrm{SL}(W)$-invariant but not $\mathrm{GL}(W)$-invariant: under $g \in \mathrm{GL}(W)$ with $\det(g) = \delta$,

$$\mathrm{vol}_W \to \delta \cdot \mathrm{vol}_W, \qquad \mathrm{vol}_{W^*} \to \delta^{-1} \cdot \mathrm{vol}_{W^*}.$$

Therefore a volume choice on W is needed to write Ω_W. The GL(7)-orbit absorbs this choice (Step 5). ■

Step 3 — The generic natural 3-form.

$$\Omega_{a,b,c} = a \cdot (\tau \wedge \mathrm{ev}) + b \cdot \mathrm{vol}_W + c \cdot \mathrm{vol}_{W^*}.$$

In coordinates $(\tau, x_1, x_2, x_3,\ y_1, y_2, y_3)$:

$$\begin{aligned} \tau \wedge \mathrm{ev} &= e_{014} + e_{025} + e_{036}, \\ \mathrm{vol}_W &= e_{123}, \\ \mathrm{vol}_{W^*} &= e_{456}. \end{aligned}$$

So: $\Omega_{a,b,c} = a\, e_{014} + a\, e_{025} + a\, e_{036} + b\, e_{123} + c\, e_{456}$.

Five terms. Volume choice on W required; GL(7)-orbit independent of choice [Step 5]. ■

Step 4 — Stability. The Hitchin bilinear form of $\Omega_{a,b,c}$ has block structure:

$$b_\Omega/6 = \begin{pmatrix} -a^3 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & abc/2 \\ \vdots & & \ddots & \\ 0 & abc/2 & \cdots & 0 \end{pmatrix}$$

with $b_{00} = -a^3$ and three 2×2 blocks $\left(\begin{smallmatrix} 0 & abc/2 \\ abc/2 & 0 \end{smallmatrix}\right)$ pairing W with W^*.

$$\det(b_\Omega) = 4374 \cdot a^9 \cdot b^6 \cdot c^6.$$

Therefore $\Omega_{a,b,c}$ is stable if and only if $abc \neq 0$. ■

Step 5 — All stable forms are in one GL(7)-orbit. For any a, b, c with $abc \neq 0$, take $g = \mathrm{diag}(\alpha, \lambda I_3, \mu I_3)$ with

$$\lambda = b^{-1/3}, \qquad \mu = c^{-1/3}, \qquad \alpha = (bc)^{1/3}/a.$$

All cube roots are real (over $\mathbb{R}$). Under pullback (covariant):

$$\begin{aligned} \tau \wedge \mathrm{ev} \text{ coefficient:} \quad & a \cdot \alpha \cdot \lambda \cdot \mu = a \cdot \frac{(bc)^{1/3}}{a} \cdot (bc)^{-1/3} = 1. \\ \mathrm{vol}_W \text{ coefficient:} \quad & b \cdot \lambda^3 = b \cdot b^{-1} = 1. \\ \mathrm{vol}_{W^*} \text{ coefficient:} \quad & c \cdot \mu^3 = c \cdot c^{-1} = 1. \end{aligned}$$

Therefore $(g^{-1})^* \Omega_{a,b,c} = \Omega_{1,1,1}$. All $\Omega_{a,b,c}$ with $abc \neq 0$ are GL(7)-equivalent. ■

Note. The volume choice on W gives a one-parameter family $\Omega_t = \tau \wedge \mathrm{ev} + t \cdot \mathrm{vol}_W + t^{-1} \cdot \mathrm{vol}_{W^*}$ (since ν^{-1} scales inversely). This is a subfamily with $a = 1$, $bc = 1$. Step 5 covers the general case.

Step 6 — Minimality of dim 7. On $W \oplus W^*$ (dim 6), the natural 3-forms are $\mathrm{vol}_W + \mathrm{vol}_{W^*}$. These have stabilizer $\mathrm{SL}(3) \times \mathrm{SL}(3)$, dim 16. Classical. The evaluation pairing $\mathrm{ev} \in W^* \otimes W$ is degree 2. To include ev in a 3-form requires one additional covector τ:

$$\tau \wedge \mathrm{ev} \in \Lambda^3(\mathbb{R} \oplus W \oplus W^*).$$

This cannot live in $\Lambda^3(W \oplus W^*)$. Therefore $\dim 7 = 1 + 3 + 3$ is minimal. ■

Conclusion. C_2 determines W, dim 3 [Proof 20]. W determines $V_W = \mathbb{R} \oplus W \oplus W^*$, dim 7 (minimal). The natural 3-form $\Omega_W = \tau \wedge \mathrm{ev} + \mathrm{vol}_W + \mathrm{vol}_{W^*}$ requires a volume choice on W. But:

The volume choice is absorbed already by $\mathrm{GL}(W)$: replacing ν by $t \cdot \nu$ gives $\Omega_t = \tau \wedge \mathrm{ev} + t \cdot \mathrm{vol}_W + t^{-1} \cdot \mathrm{vol}_{W^*}$, and $g = \mathrm{diag}(1, t^{-1/3} I_3, t^{1/3} I_3) \in \mathrm{GL}(W) \subset \mathrm{GL}(7)$ sends Ω_t back to Ω_1. No full GL(7) needed.

Step 5 proves a stronger statement: all $\Omega_{a,b,c}$ with $abc \neq 0$ (including $a \neq 1$, $bc \neq 1$) are GL(7)-equivalent. This goes beyond volume choice—it covers deformations that break W/W^* duality.

C_2 determines the GL(7)-orbit. The representative requires a volume choice. The choice is absorbed by $\mathrm{GL}(W)$.

[Dependencies: Proof 20. Algebraic—no computation needed.
Note: "functorial" means functorial in oriented W ($\mathrm{SL}(W)$), not in bare W ($\mathrm{GL}(W)$). The GL(7)-orbit is independent of the orientation/volume choice.]